Online Appendix

Quality is in the Eye of the Beholder:
Taste Projection in Markets with Observational Learning

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This Online Appendix is organized as follows. Section B considers the key effects of projection under richer signal structures. Section C considers additional results from the dynamic model with an arbitrary number of periods, including results on optimal monopoly pricing.

B Alternative Signal Structures

In this section, we show that our key comparative statics from the main model emerge in settings with richer heterogeneity in private information. We also note a few additional implications that emerge in these settings.

B.1 Fully-Heterogeneous Private Signals

We first consider the case in which each agent receives a conditionally independent private signal correlated with ω. We show that a projector's inferred quality upon observing the aggregate quantity demanded by these privately informed agents is still: (i) negatively related to her taste; and (ii) positively related to the price that predecessors paid. We will show this in a two-period model similar to Section 4.

As in the main text, suppose that individuals share a common prior over ω with support $\mathbb{R}$. In each generation $n = 1, 2$, individual i observes the realization of a private signal $S_{i,n}$ that is correlated with ω. We assume that, conditional on ω, signals are i.i.d. across all individuals in both periods, and that no signal realization perfectly reveals ω. Let $Z_{i,n} \equiv \mathbb{E}[\omega|S_{i,n}]$ denote a consumer's "private belief"—their expected quality conditional on their signal and the prior. We work directly with the distribution of $Z_{i,n}$ conditional on ω rather than conditional distributions over signals. As such, let $Z(\omega)$ denote the random variable representing individuals' private beliefs conditional on ω. We assume that $Z(\omega)$ can be expressed as $Z(\omega) = m(\omega) + Y$ for some strictly increasing function

m and a random variable Y that is independent of ω (and T) and has a log-concave density.[1] This implies that consumers' interim valuations for the good in period 1 are distributed according to $V(\omega) \equiv m(\omega) + Y + T$. Let $H(\cdot;\omega)$ denote the CDF of $V(\omega)$. In period 1, individuals act on their private signals alone. Thus, the demand function in period 1 is $D_1(p;\omega) \equiv 1 - H(p_1;\omega)$.

Fixing the true quality ω, we are interested in the quality inferred by consumers in period 2 upon observing $d_1 = D(p_1;\omega)$ and price p_1. Let $\hat{\omega}(t;p_1)$ denote the quality inferred by a consumer with taste t.

Proposition B.1 (Comparative Statics in the Heterogeneous-Signal Model)**.** *Consider the signal structure of Section B.1. Fix ω, and consider any p_1 such that demand in period 1 is interior (i.e., $d_1 \in (0,1)$). For any $\alpha > 0$, the inferred quality of a projector with type t who observes d_1 is: (i) decreasing in t (ii) increasing in p_1.*

The proof, presented below, follows a similar logic to the graphical argument in Figure 1. Since a projector thinks interim valuations are less dispersed than they truly are, her perceived demand curve intersects the true demand curve at a point where the perceived demand curve has a greater price elasticity. Thus, to explain a market outcome at a higher price, the projector must consider a demand curve that is shifted outward relative to the initial perceived demand. This outward shift corresponds to a higher perceived quality. The key difference between this case and the one considered in the main text is that the observed quantity demanded now results from both variation in consumers' tastes and variation in their signals. We therefore make use of results on the "dispersion ordering" of convolutions of log-concave random variables to prove that, even when consumers' have disperse private information, the perceived and true demand curves continue to obey a single-crossing property crucial to the logic depicted in Figure 1.

Proof of Proposition B.1. Fix ω, and consider any p_1 such that the quantity demanded in period 1 is interior (i.e., $d_1 \in (0,1)$). We examine how $\hat{\omega}(t;p_1)$ varies in t and p_1. Note that $\hat{\omega}(t;p_1)$ is the value of $\hat{\omega}$ that solves $\widehat{D}_1(p_1;\hat{\omega}|t) = D_1(p_1;\omega)$, where $\widehat{D}_1(p_1;\hat{\omega}|t)$ is type t's misperceived demand function: $\widehat{D}_1(p_1;\hat{\omega}|t) = 1 - \widehat{H}(p;\hat{\omega}|t)$, and $\widehat{H}(\cdot;\hat{\omega}|t)$ is the CDF of $\widehat{V}(\hat{\omega}|t) \equiv m(\hat{\omega}) + Y + \widehat{T}(t)$. Hence $\hat{\omega}(t;p_1)$ is the value of $\hat{\omega}$ that solves $L(\hat{\omega};t,p_1) \equiv \widehat{D}_1(p_1;\hat{\omega}|t) - D_1(p_1;\omega) = 0$.

Part 1: The Effect of t on Perceived Quality. By the Implicit Function Theorem (IFT):

$$\frac{\partial \hat{\omega}(t;p_1)}{\partial t} = -\frac{\partial L(\hat{\omega};t,p_1)}{\partial t}\left(\frac{\partial L(\hat{\omega};t,p_1)}{\partial \hat{\omega}}\right)^{-1}\Bigg|_{\hat{\omega}=\hat{\omega}(t;p_1)}. \tag{B.1}$$

Notice that, for any p_1 that generates interior demand and any t, $\frac{\partial}{\partial \hat{\omega}} L(\hat{\omega};t,p_1) = \frac{\partial}{\partial \hat{\omega}} \widehat{D}_1(p_1;\hat{\omega}|t) > 0$ given our mild assumption that demand is increasing in quality (i.e., m is a strictly increasing

[1] This structure nests the familiar Gaussian structure noted in the main text, but is also more general.

function). Thus

$$\operatorname{sgn}\left(\frac{\partial\hat{\omega}(t;p_1)}{\partial t}\right)=\operatorname{sgn}\left(-\frac{\partial L(\hat{\omega};t,p_1)}{\partial t}\bigg|_{\hat{\omega}=\hat{\omega}(t;p_1)}\right). \tag{B.2}$$

Note that

$$-\frac{\partial L(\hat{\omega};t,p_1)}{\partial t}=-\frac{\partial}{\partial t}\widehat{D}_1(p_1;\hat{\omega}|t)<0. \tag{B.3}$$

This follows from the fact that $t'>t$ implies that $\widehat{V}(\hat{\omega}|t')$ first-order stochastically dominates $\widehat{V}(\hat{\omega}|t)$ since in this case $\widehat{T}(t')$ first-order stochastically dominates $\widehat{T}(t)$; accordingly, $\widehat{H}(p;\hat{\omega}|t)$ is decreasing in t and thus $\widehat{D}_1(p;\hat{\omega}|t)$ is increasing in t.

Part 2: The Effect of p *on Perceived Quality.* Invoking the IFT again, the discussion following (B.1) implies that

$$\operatorname{sgn}\left(\frac{\partial\hat{\omega}(t;p)}{\partial p}\right)=\operatorname{sgn}\left(-\frac{\partial L(\hat{\omega};p)}{\partial p}\bigg|_{\hat{\omega}=\hat{\omega}(t;p_1)}\right). \tag{B.4}$$

Note that

$$-\frac{\partial L(\hat{\omega};p)}{\partial p}=\frac{\partial}{\partial p}D_1(p;\omega)-\frac{\partial}{\partial p}\widehat{D}_1(p;\hat{\omega}|t). \tag{B.5}$$

With downward-sloping demand functions, the previous expression is positive when evaluated at $\hat{\omega}(t;p_1)$ if and only if

$$\left|\frac{\partial}{\partial p}D_1(p_1;\omega)\right|<\left|\frac{\partial}{\partial p}\widehat{D}_1(p_1;\hat{\omega}(t;p_1)|t)\right|; \tag{B.6}$$

that is, if and only if the perceived demand function is locally more price sensitive at the original market outcome than the true demand function.

Since $\hat{\omega}(t;p_1)$ is a state in which type t's perceived demand curve intersects the true demand curve at the observed market outcome (d_1,p_1) (i.e., $\widehat{D}_1(p_1;\hat{\omega}(t;p)|t)=d_1=D_1(p_1;\omega)$), a sufficient condition for Condition (B.6) is that for any arbitrary $\hat{\omega}$, $\widehat{D}_1(\cdot;\hat{\omega}|t)$ crosses $D_1(\cdot;\omega)$ at most once and does so from above. That is, there exists at most one price p^* such that $\widehat{D}_1(p^*;\hat{\omega}|t)=D_1(p^*;\omega)$, and p^* is such that $\widehat{D}_1(p;\hat{\omega}|t)<D_1(p;\omega)$ for all $p>p^*$ and $\widehat{D}_1(p;\hat{\omega}|t)>D_1(p;\omega)$ for all $p<p^*$. (Note that the demand curves in Figure 1 are drawn, as usual, with p on the y-axis; from that perspective, the previous condition implies that the perceived demand curve crosses the true one from below.)

To complete the proof, we prove the sufficient condition above: for any arbitrary $\hat{\omega}$ and t, there exists at most one price p^* such that $\widehat{D}_1(p^*;\hat{\omega}|t)=D_1(p^*;\omega)$, and p^* is such that $\widehat{D}_1(p;\hat{\omega}|t)<D_1(p;\omega)$ for all $p>p^*$ and $\widehat{D}_1(p;\hat{\omega}|t)>D_1(p;\omega)$ for all $p<p^*$. Given that $D_1(p;\omega)=1-H(p;\hat{\omega})$ and $\widehat{D}_1(p;\hat{\omega}|t)=1-\widehat{H}(p;\hat{\omega}|t)$, it suffices to show that $\widehat{H}(p|\hat{\omega};t)$ crosses $H(p|\omega)$ at most once and does so from below (i.e., there exists at most one price p^* such that $\widehat{H}(p|\hat{\omega};t)<H(p;\omega)$ if $p<p^*$ and $\widehat{H}(p|\hat{\omega};t)>H(p;\omega)$ if $p>p^*$).

We prove this using the concept of dispersive order defined by Shaked (1982) and Shaked and Shanthikumar (2007). For any two arbitrary random variables X and Y with CDFs F_X and F_Y, we say that X is less dispersed than Y, denoted $X \leq_{disp} Y$, if $F_X^{-1}(b) - F_X^{-1}(a) \leq F_Y^{-1}(b) - F_Y^{-1}(a)$ whenever $0 \leq a \leq b \leq 1$. By Theorem 2.1 of Shaked (1982), $X \leq_{disp} Y$ iff F_X crosses F_Y at most once and does so from below. Thus, it suffices to show that $\widehat{V}(\hat{\omega}; t) \leq_{disp} V(\omega)$, which is equivalent to $\widehat{T}(t) + Z(\hat{\omega}) \leq_{disp} T + Z(\omega)$. Since $Z(\omega) = m(\omega) + Y$, the previous condition is equivalent to $\widehat{T}(t) + m(\hat{\omega}) + Y \leq_{disp} T + m(\omega) + Y$, where $m(\hat{\omega})$ and $m(\omega)$ are constants given that we are conditioning on ω and $\hat{\omega}$. As noted in Comment 3.B.2 of Shaked and Shanthikumar (2007), the order $\leq_{disp}$ is location invariant, meaning that $\widehat{T}(t) + m(\hat{\omega}) + Y \leq_{disp} T + m(\omega) + Y \Leftrightarrow \widehat{T}(t) + Y \leq_{disp} T + Y$. Since Y has a log-concave density and is independent of T and $\widehat{T}(t)$, Theorem 3.B.8 of Shaked and Shanthikumar (2007) implies that $\widehat{T}(t) + Y \leq_{disp} T + Y$ if $\widehat{T}(t) \leq_{disp} T$. Thus, to complete the proof it suffices to show that $\widehat{T}(t) \leq_{disp} T$. Again by Theorem 2.1 in Shaked (1982), this holds so long as $\widehat{F}(\cdot|t)$ crosses F only once and does so from below. This is true by Part 4 of Observation 1, completing the proof. ■

B.2 Heterogeneous Signals Across Periods

In this section, we consider a structure in which each generation of consumers observes a distinct signal. All consumers in each Generation n observe the same signal realization, which we denote by s_n. We assume that s_n is i.i.d. for all n. Furthermore, s_n is "quasi-public": it is observed by all agents within Generation n, but not by agents in any other generation.[2] As in the main text (and the previous appendix section), we again show that the perceived quality of each agent in each Generation $n \geq 2$ is: (i) negatively related to their taste; and (ii) positively related to the price that predecessors paid.

Setup. Agents in Generation n attempt to infer the posterior beliefs of agents in period $n-1$ from their quantity demanded. If agents are rational, then all agents in each generation hold a common expectation over ω. Let $\tilde{\omega}_{n-1}$ denote this rational expectation among Generation $n-1$ for $n \geq 2$. Agents in Generation n can then perfectly extract $\tilde{\omega}_{n-1}$ from the observed market coverage in Generation $n-1$ (assuming this value is interior).

To make matters concrete, we consider the familiar Gaussian information structure: $\omega \sim N(\bar{\omega}_0, \rho^2)$, and $s_n \sim N(\omega, \eta^2)$. Rational updating then takes the form

$$\tilde{\omega}_n = \gamma_n s_n + (1-\gamma_n)\tilde{\omega}_{n-1}, \quad \text{where} \quad \gamma_n = \frac{1}{n + \eta^2/\rho^2}. \tag{B.7}$$

As the updating process in B.7 suggests, a rational Generation n will combine their own signal, s_n, with the inferred posterior belief of Generation $n-1$, $\tilde{\omega}_{n-1}$, to reach their posterior estimate of ω.

With projection, an agent in Generation n *thinks* he can perfectly extract the posterior expectation

[2] If generations consisted of a single agent, this structure would resemble the canonical sequential herding model (e.g., Banerjee, 1992; Bikhchandani et al., 1992; Smith and Sørensen, 2000).

of ω held by the previous generation, but does so incorrectly. As usual, his incorrect inference will depend on his taste, t. Denote this (mis)extracted value of $\tilde{\omega}_{n-1}$ by $\hat{\omega}_{n-1}(t)$. The projector will then use B.7 to form a posterior estimate of $\gamma_n s_n + (1-\gamma_n)\hat{\omega}_{n-1}(t)$. Below, we analyze how projectors' beliefs evolve within this structure.

We first consider how beliefs evolve within the first few periods. For simplicity, we normalize $\bar{\omega}_0 = 0$. Since Generation 1 does not observe others, their is no scope for mislearning in period 1. Hence, agents in Generation 1 share a common (rational) estimate of ω equal to $\tilde{\omega}_1 = \gamma_1 s_1$. Thus, an agent buys iff $\tilde{\omega}_1 + t_i \geq p_1 \Leftrightarrow t_i \geq p_1 - \tilde{\omega}_1$, and hence demand in period 1 is

$$D_1(p_1; \tilde{\omega}_1) = 1 - F(p_1 - \tilde{\omega}_1). \tag{B.8}$$

Distorted Beliefs in Generation 2. An agent in Generation 2 with taste t thinks that, conditional on Generation 1 holding a posterior expectation of $\hat{\omega}$, their demand is given by

$$\widehat{D}_1(p_1; \hat{\omega}|t) = 1 - \widehat{F}(p_1 - \hat{\omega}|t) = 1 - F\left(\frac{p_1 - \hat{\omega} - \alpha t}{1-\alpha}\right). \tag{B.9}$$

This agent wrongly infers that the posterior expectation in Generation 1 is the value of $\hat{\omega}$ that solves $D(p_1; \tilde{\omega}_1) = \widehat{D}(p_1, \hat{\omega}|t)$, which we denote by $\hat{\omega}_1(t)$. Hence,

$$\hat{\omega}_1(t) = (1-\alpha)\tilde{\omega}_1 + \alpha(p_1 - t). \tag{B.10}$$

This misperception is identical to the one formed by agents in Generation 2 of the baseline model in the main text (see Equation 10). Furthermore, given that $\tilde{\omega}_1 = \gamma_1 s_1$, the preceding equation implies that an agent with taste t misinfers the signal to be

$$\hat{s}_1(t) = (1-\alpha)s_1 + \frac{1}{\gamma_1}\alpha(p_1 - t). \tag{B.11}$$

An immediate implication of (B.10) and (B.11) is that, under projection, an observer underweights the true information of the previous generation. Moreover, they wrongly put weight on irrelevant factors (i.e., the price and their own taste), and this erroneous weight is larger when signals are less precise relative to the prior (i.e., when γ_1 is smaller). There is a straightforward intuition for this. A projector will, on average, observe a level of demand that deviates from their initial expectations since they incorrectly predict demand conditional on the signal. They attribute this deviation to the value of s_1. Thus, when a projector anticipates that the signal will have little effect on predecessors' beliefs (i.e.,. γ_1 is small), they require a more extreme value of s_1 to rationalize the deviation between the observed demand and their biased predictions.

Now consider demand in Generation 2. An agent with taste t forms an expectation of ω based on

s_2 and $\hat{\omega}_1(t)$ equal to $\mathbb{E}[\omega|s_2, \hat{\omega}_1(t)] = \gamma_2 s_2 + (1-\gamma_2)\hat{\omega}_1(t)$. Using the expression for $\hat{\omega}_1(t)$ above, the expected valuation of an agent in Generation 2 with taste t is

$$\mathbb{E}[u(\omega,t)|s_2,\hat{\omega}_1(t)] = \gamma_2 s_2 + (1-\gamma_2)\Big((1-\alpha)\tilde{\omega}_1 + \alpha p_1\Big) + \Big(1-\alpha(1-\gamma_2)\Big)t. \tag{B.12}$$

Let $\hat{v}_2(t)$ denote the expected valuation in (B.12). Similar to the approach in the main text, we can write this perceived valuation in terms of a taste-independent component, denoted by $\bar{\omega}_2$, where

$$\bar{\omega}_2 \equiv \gamma_2 s_2 + (1-\gamma_2)\Big((1-\alpha)\tilde{\omega}_1 + \alpha p_1\Big). \tag{B.13}$$

In the rational model (i.e., $\alpha = 0$), $\bar{\omega}_2$ reduces to $\tilde{\omega}_2$—the rational expectation of ω given (s_1, s_2). Given (B.13), we can write perceived valuations in Generation 2 as $\hat{v}_2(t) = \bar{\omega}_2 + \beta_2 t$, where $\beta_2 \equiv 1-\alpha(1-\gamma_2)$.

The Evolution of Beliefs. In fact, the perceived valuations of consumers in *all* Generations $n \geq 2$ can be expressed as $\hat{v}_n(t) = \bar{\omega}_n + \beta_n t$ where $\bar{\omega}_n$ is independent of tastes. Thus, the dynamics of the model are described by the evolution of the sequences of $(\bar{\omega}_n)$ and (β_n).

To verify for this claim, suppose that, as in Generation 2, the perceived valuations of agents in any Generation $n > 2$ are given by $\hat{v}_n(t) = \bar{\omega}_n + \beta_n t$. The demand in period $n \geq 2$ is then

$$D_n(p_n; \bar{\omega}_n) \equiv 1 - F\left(\frac{1}{\beta_n}(p_n - \bar{\omega}_n)\right). \tag{B.14}$$

A projecting agent in Generation $n+1$ with taste t thinks that agents in Generation n share a common expectation of ω, denoted $\hat{\omega}$, and thus have a demand given by

$$\widehat{D}_n(p_n; \hat{\omega}|t) = 1 - \widehat{F}(p_n - \hat{\omega}|t) = 1 - F\left(\frac{p_n - \hat{\omega} - \alpha t}{1-\alpha}\right). \tag{B.15}$$

The agent thus infers that the posterior expectation of Generation n is the value of $\hat{\omega}$ that equates (B.14) and (B.15), yielding

$$\hat{\omega}_n(t) = \left(\frac{1-\alpha}{\beta_n}\right)\bar{\omega}_n + \left(1 - \frac{1-\alpha}{\beta_n}\right)p_n - \alpha t. \tag{B.16}$$

Thus, the updated expectation of ω for an agent with taste t in Generation $n+1$ is

$$\mathbb{E}[\omega|s_{n+1}, \hat{\omega}_n(t)] = \gamma_{n+1}s_{n+1} + (1-\gamma_{n+1})\left[\left(\frac{1-\alpha}{\beta_n}\right)\bar{\omega}_n + \left(1-\frac{1-\alpha}{\beta_n}\right)p_n - \alpha t\right]. \tag{B.17}$$

This agent's total perceived valuation is $\hat{v}_{n+1}(t) = \mathbb{E}[\omega | s_{n+1}, \hat{\omega}_n(t)] + t$; hence,

$$\hat{v}_{n+1}(t) = \underbrace{\gamma_{n+1}s_{n+1} + (1-\gamma_{n+1})\left[\left(\frac{1-\alpha}{\beta_n}\right)\bar{\omega}_n + \left(1 - \frac{1-\alpha}{\beta_n}\right)p_n\right]}_{\equiv \bar{\omega}_{n+1}} + \underbrace{\Big(1-\alpha(1-\gamma_{n+1})\Big)}_{\equiv \beta_{n+1}} t.$$

This reveals how (β_n) and $(\bar{\omega}_n)$ evolve:

$$\beta_{n+1} = 1 - \alpha(1-\gamma_{n+1}), \tag{B.18}$$

$$\bar{\omega}_{n+1} = \gamma_{n+1}s_{n+1} + (1-\gamma_{n+1})\left[\left(\frac{1-\alpha}{\beta_n}\right)\bar{\omega}_n + \left(1 - \frac{1-\alpha}{\beta_n}\right)p_n\right]. \tag{B.19}$$

Thus, for all $n \geq 2$, the perceived valuations of consumers in period n are given by $\hat{v}_n(t) = \bar{\omega}_n + \beta_n t$, where β_n and $\bar{\omega}_n$ follow the processes in (B.18) and (B.19), respectively, starting from the initial conditions of $\beta_1 = 1$ and $\bar{\omega}_1 = \tilde{\omega}_1 = \gamma_1 s_1$. Furthermore, the quantity demanded in each period n is given by $d_n = D_n(p_n; \bar{\omega}_n)$ as in (B.14).[3]

There are a few features of this process worth noting. First, since γ_n is monotonically decreasing in n with $\lim_{n\to\infty} \gamma_n = 0$, it follows that β_n monotonically decreases from 1 and converges to $1-\alpha$. Thus, in every period, a consumer's perceived valuation puts too little (yet positive) weight on his own taste. In the limit, this diminished weight is equal to $1-\alpha$. This is identical to our results in both the static and dynamic cases of our model in the main text. See, for instance, the discussion preceding Proposition 2.

Additionally, since $\beta_n \in (1-\alpha, 1)$ for all n, the term $(1-\alpha)/\beta_n$ that appears in the transition equation for $(\bar{\omega}_n)$ must take a value in $(0, 1)$. Thus, the term in square brackets in Equation (B.19) is a convex combination of $\bar{\omega}_n$ and p_n, implying that the aggregate biased belief in each period n is strictly increasing in the price faced by the previous generation. Furthermore, the weight on $\bar{\omega}_n$ (i.e., $(1-\alpha)/\beta_n$) converges to 1 as $n \to \infty$, and thus the effect of the preceding price on current beliefs diminishes over time.

Finally, we can use Equation (B.19) to write the beliefs of the current generation in terms of the entire history of signals and prices. Toward that end, let $\lambda_n \equiv (1-\alpha)/\beta_n \in (0, 1)$. For all $k = 1, 2, \dots$ and all $n \geq k+2$, define $a_k^n = \prod_{j=k+1}^{n-1} \lambda_j$. We then have:

$$\bar{\omega}_n = \gamma_n s_n + (1-\alpha)\gamma_n\left(\frac{1}{\beta_{n-1}}s_{n-1} + \sum_{k=1}^{n-2}\frac{a_k^n}{\beta_k}s_k\right) + \alpha\gamma_n\left(\frac{1}{\beta_{n-1}}p_{n-1} + \sum_{k=2}^{n-2}\frac{a_k^n}{\beta_k}p_k + \frac{a_1^n}{\gamma_1}p_1\right). \tag{B.20}$$

[3] Note that the transition equations in (B.18) and (B.19) characterize the process in the case where the quantity demanded in each period prior to $n+1$ is interior (i.e., $d_k \in (0, 1)$ for $k \leq n$).

The key implications of this expression are that aggregate biased beliefs put too little weight on predecessors' signals and instead erroneously put positive weight on all past prices.

The next result summarizes some of the points above, emphasizing that the comparative statics in our baseline model of the main text continue to hold within this richer signal structure.

Proposition B.2 (Comparative Statics in the Quasi-Public-Signal Model)**.** *Consider the signal structure of Section B.2. Beliefs and valuations in each period n follow the process described in* (B.19) *so long as demand remains interior (i.e., $d_k \in (0,1)$ for all $k < n$). In this case, the perceived quality of each agent in each period $n \geq 2$ is decreasing in their private value and increasing in all previous prices.*

C Additional Results on the Dynamic Model with an Arbitrary Horizon

In this section, we consider some additional results from the dynamic model (Section 4) with an arbitrary number of periods, N. Section C.1 considers the evolution of beliefs, and Section C.2 considers dynamic monopoly pricing.

C.1 Details on Belief Dynamics with an Arbitrary Number of Periods

This section supplements the discussion in Section 4.3 in the main text by providing additional details on how beliefs and aggregate behavior evolve over time. First, consider inferences among Generation 3. Generation 3 forms their quality expectations based on the quantity demanded in period 2, which is given by (12) in the main text. While misinference among Generation 2 stemmed directly from misunderstanding others' tastes (i.e., an error in first-order beliefs), the misinference among Generation 3 also includes a "social misinference" effect stemming from naivete about others' projection. Namely, individuals neglect that their predecessors failed to reach consistent beliefs. Since uninformed consumers expect to extract s form their predecessors' behavior, an individual in period 3 accordingly thinks that the uninformed consumers in period 2 consistently and correctly inferred s and are thus now informed. This presumption is false: projectors in period 2 draw biased, taste-dependent beliefs (as in Equation 10). Nevertheless, an uninformed projector in Generation 3 with taste t thinks period-2 demand is determined by the function $\widehat{D}^I(p_2;\hat{\omega}|t)$ in (9)—she does not realize that it derives from a composition of demand functions as in (12). This observer then infers a value of $\hat{\omega}$ that solves $d_2 = \widehat{D}^I(p_2;\hat{\omega}|t)$, which we denote by $\hat{\omega}_3(t)$. As with Generation 2, if we let $\bar{\omega}_3$ denote the taste-independent part of $\hat{\omega}_3(t)$, then we can write $\hat{\omega}_3(t) = \bar{\omega}_3 - \alpha t$. Aggregate demand among Generation 3 then follows the same form as Generation 2: $d_3 = D(p_3;\bar{\omega}_3,\bar{\omega}(s))$ where D is as defined in (12).

A similar logic unfolds in each period $n \geq 2$. Indeed, Part 1 of Proposition 6 shows that the perceived quality among uninformed agents in Generation n can be written in terms of a taste-independent component, denoted by $\bar{\omega}_n$. For the remainder of this appendix, we refer to $\bar{\omega}_n$ as the *aggregate biased belief* in period n.

Despite a continuum of types forming distinct beliefs from each observation, the previous result implies that we can account for this infinite-dimensional process by studying the evolution of the unidimensional sequence, $(\bar{\omega}_n)$. Since this sequence describes the path of uninformed consumers' beliefs, the quantity demanded in each period n, d_n, is given by

$$D(p_n; \bar{\omega}_n, \bar{\omega}(s)) = \underbrace{\lambda\big[1 - F(p_n - \bar{\omega}(s))\big]}_{\text{Informed Demand}} + \underbrace{(1-\lambda)\left[1 - F\left(\frac{p_n - \bar{\omega}_n}{1-\alpha}\right)\right]}_{\text{Uninformed Demand}}. \tag{C.1}$$

However, an uninformed consumer in period $n+1$ thinks that d_n is determined by

$$\widehat{D}(p_n; \bar{\omega}_{n+1}) \equiv 1 - F\left(\frac{p_n - \bar{\omega}_{n+1}}{1-\alpha}\right).^{4}$$

Furthermore, $\bar{\omega}_{n+1}$ must be consistent with d_n for all $n \geq 2$; that is, $d_n = \widehat{D}(p_n; \bar{\omega}_{n+1})$. Hence, the law of motion describing the process $(\bar{\omega}_n)$ is characterized by the condition

$$\widehat{D}(p_n; \bar{\omega}_{n+1}) = D(p_n; \bar{\omega}_n, \bar{\omega}(s)), \tag{C.2}$$

starting from the initial condition of $\bar{\omega}_2 = (1-\alpha)\bar{\omega}(s) + \alpha p_1$.

Before turning to the optimal price path given this belief process, we describe outcomes under two benchmark scenarios: (i) a constant price, and (ii) a single change in price. First, if the price is fixed at p (e.g,. the market is in a competitive equilibrium or other frictions mandate a fixed price), then $\bar{\omega}_n = \bar{\omega}_2$ for all $n > 2$. Thus, beliefs remain constant over time, and the quantity demanded in each period matches the rational benchmark at price p. Intuitively, since the type in Generation 2 who learns correctly has a private value equal to the rational marginal type, this type will again be marginal given that the price is constant. Hence, Generation 2 demands the same quantity as Generation 1. Since Generation 3 then observes the same quantity as Generation 2 did, they draw the same inference. This result reflects the notion that our dynamic process can be viewed as starting from the steady-state: when the price stays constant, the system remains fixed.

On the other hand, when the price changes, aggregate demand will initially overreact and then slowly converge back to the rational level given the new price. The logic is similar to the reason why demand among the uninformed in Generation 2 is excessively sensitive to p_2 (e.g., the discussion

[4] More precisely, an uninformed consumer in period $n+1$ with taste t thinks d_n is determined by $\widehat{D}^I(p_n; \hat{\omega}_{n+1}(t)|t)$ as in (9). Applying the fact that $\hat{\omega}_{n+1}(t) = \bar{\omega}_{n+1} - \alpha t$ yields the expression here.

around Figure 2 in the main text). However, the result below shows that the overreaction to price changes in period 2 spills onto later generations as well. For instance, suppose the price permanently drops in period 2. All uninformed types with a private value below the marginal type from Generation 1 overestimate ω; hence, relative to the rational benchmark, a larger measure of those who were originally submarginal buy once the price drops. A similar overreaction occurs if the price instead increases. (Proofs for all results in Section C are presented in Section C.3.)

Proposition C.1. *Suppose there exists a period $n^* \geq 1$ such that $p_n = p$ for $n \leq n^*$, and $p_n = \tilde{p} \neq p$ for all $n > n^*$. Consider s such that both (p, s) and $(\tilde{p}, s)$ admit interior demand, and let $\tilde{d}$ denote the quantity demanded at price $\tilde{p}$ under rational learning. Then, for any $\alpha > 0$ the following hold:*

1. Initial Overreaction: *If $\tilde{p} > p$, then $d_n < \tilde{d}$ for all $n > n^*$. If instead $\tilde{p} < p$, then $d_n > \tilde{d}$ for all $n > n^*$.*
2. Convergence to Rational Equilibrium: *$|d_n - \tilde{d}|$ is decreasing in n and $\lim_{n\to\infty} |d_n - \tilde{d}| = 0$.*

C.2 Dynamic Monopoly Pricing with an Arbitrary Number of Periods

Building on our analysis from Section 4.2, we consider the optimal dynamic price profile for a monopolist facing an arbitrary number of periods. In particular, we show how our declining-price result extends beyond $N = 2$ for the case of uniformly distributed tastes: the initial price is inflated above the static monopoly price, and prices gradually decline thereafter. This result follows from a novel trade-off the seller faces in any given period (aside from the first or last). On the one hand, keeping the price high and restraining current sales helps to maintain inflated beliefs further into the future. On the other hand, lowering the current price allows the seller to reap high current sales by exploiting the inflated beliefs generated by high prices in previous periods. This intertemporal trade-off results in an optimal price path that gradually declines.

The seller chooses a sequence of prices $(p_1, \ldots, p_N)$ to maximize

$$\Pi \equiv p_1 D^I(p_1; \bar{\omega}(s)) + \sum_{n=2}^{N} p_n D(p_n; \bar{\omega}_n, \bar{\omega}(s)) \tag{C.3}$$

subject to the dynamic constraint in (C.2) for all $n \geq 2$, where $D(p_n; \bar{\omega}_n, \bar{\omega}(s))$ is given by (C.1). We focus on the case in which private values are uniformly distributed over $[\underline{t}, \bar{t}]$; see the discussion in Section 4.2 around Figure 4 for details on this case. Additionally, we restrict attention to interior cases where it is never optimal to serve the lowest type (which amounts to assuming $\underline{t}$ is sufficiently low).[5] Equation (C.2) implies that the aggregate biased belief—that is, the taste-independent com-

[5] With uniform tastes, our usual assumption that (p^M, s) admits interior demand is equivalent to $\bar{\omega}(s) + \bar{t} > 0$ and $\bar{\omega}(s) < \bar{t} - 2\underline{t}$. It is never optimal to serve the lowest projecting type if we also have $(1 - \alpha)\bar{\omega}(s) + \alpha\bar{p} < \bar{t} - 2\underline{t}$.

ponent of beliefs, $\bar{\omega}_n$, from Part 1 of Proposition 6—evolves according to

$$\bar{\omega}_{n+1} = \lambda\big[(1-\alpha)\bar{\omega}(s) + \alpha p_n\big] + (1-\lambda)\bar{\omega}_n. \tag{C.4}$$

The following lemma provides an explicit expression of this aggregate belief for the case of uniformly distributed tastes.

Lemma C.1. *Suppose (p_k, s) admits interior demand for all $k \leq n$. The aggregate belief in period n is $\bar{\omega}_n = (1-\alpha)\bar{\omega}(s) + \alpha\tilde{p}^{n-1}$, where $\tilde{p}^{n-1}$ is a weighted average of past prices:*

$$\tilde{p}^{n-1} \equiv (1-\lambda)^{n-2} p_1 + \sum_{k=2}^{n-1} \lambda(1-\lambda)^{n-1-k} p_k. \tag{C.5}$$

Since the weights on all past prices in (C.5) sum to one (by virtue of being a weighted average), the overall effect of past prices on $\bar{\omega}_n$ is always equal to α. Notably, however, more recent prices have a bigger impact on the current belief than earlier ones.

The "stock variable" $\tilde{p}^{n-1}$ captures the sway of past prices on current beliefs. As such, it is convenient to re-write the demand of Generation n in terms of $\tilde{p}^{n-1}$ rather than $\bar{\omega}_n$. From (C.1) and Lemma C.1, demand in period n as a function of each previous price is

$$D(p_n; \tilde{p}^{n-1}, \bar{\omega}(s)) = \frac{(1-\alpha)\left(\bar{t} + \bar{\omega}(s)\right) + \alpha(1-\lambda)\tilde{p}^{n-1} - (1-\lambda\alpha)p_n}{(1-\alpha)(\bar{t} - \underline{t})}. \tag{C.6}$$

Given the objective function in (C.3), we then arrive at the following first-order condition for the price in a non-terminal period $n \geq 2$:

$$p_n = \frac{1}{1-\lambda\alpha}\left((1-\alpha)p^M + \frac{\alpha(1-\lambda)}{2}\left[\tilde{p}^{n-1} + \sum_{k=n+1}^{N} p_k \frac{\partial \tilde{p}^{k-1}}{\partial p_n}\right]\right), \tag{C.7}$$

where we've used the fact that $p^M = \left(\bar{t} + \bar{\omega}(s)\right)/2$ when (p^M, s) admits interior demand. The term in squared brackets in Equation (C.7) highlights the intertemporal incentives in pricing. Namely, the seller has a greater incentive to inflate the current price in order to manipulate future consumers' beliefs when the current period is earlier in the horizon, and thus influences a greater number of subsequent generations. This leads to an optimal price path that declines over time.

Proposition C.2. *Consider the setup of Section C.2, and suppose that (p^M, s) admits interior demand. For any $\alpha > 0$:*

1. *The initial price is inflated: $p_1^* > p^M$.*

2. *The optimal price path is declining: For all $n \geq 2$, we have $p_n^* < p_{n-1}^*$.*

As discussed above, this result follows from the seller balancing the trade-off between manipulating the beliefs of future consumers by maintaining a high current price versus exploiting consumers' current beliefs by undercutting the previous price. Figure C.1 provides an example of the optimal price path for $N = 20$ for different degrees of projection. Intuitively, the extent to which prices deviate from the static monopoly price increases when α is high, since in this case prices have more sway on beliefs. Although it's not captured in Figure C.1, a similar intuition holds as λ decreases: deviating from the monopoly price is less costly when there are fewer informed agents.

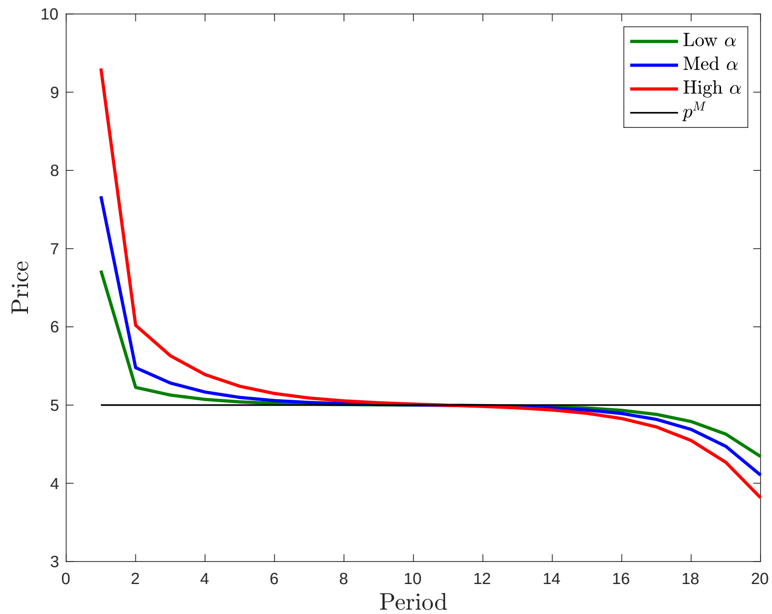

Figure C.1: *Example price path for $N = 20$ for various degrees of projection. The example assumes $\bar{t} = 10$ and $\bar{\omega}(s) = 0$.*

This declining price path also has natural implications for the evolution of aggregate beliefs and demand. Since the current aggregate belief is a convex combination of the previous belief and price, a declining price path implies that beliefs also decline over time: later generations of consumers perceive a lower quality, on average, than earlier generations. Additionally, the quantity demanded in periods with distorted beliefs (i.e., for period 2 onward) is "U-shaped": the inflated price in the first period leads Generation 2 to demand an aggregate quantity above the rational benchmark. However, as the price levels off near the rational monopoly price, the aggregate demand converges to the rational monopoly level.[6] Finally, near the end of the horizon—once there is little remaining incentive to maintain high prices to manipulate future generations—the seller will lower the price below p^M since the market demand has become more elastic, which again leads to significantly more sales than the rational monopoly benchmark.

[6] This reflects the fact that aggregate demand in the steady state of our model matches the aggregate demand under rational learning (Section 3). Hence, when the price is near constant for many periods, the resulting quantity demanded converges to the rational level given that (near) constant price; see Proposition C.1.

C.3 Proofs of Results in Appendix C

Proof of Proposition C.1. *Part 1: Initial Overreaction.* We will focus on the case with $\tilde{p} < p$; the case with $\tilde{p} > p$ is analogous and thus omitted.

Step 1: Quantity demanded is constant prior to the price change. Suppose $n^* \geq 2$. For ease of exposition, let $d^I \equiv D^I(p; \bar{\omega}(s))$ and $\tilde{d}^I \equiv D^I(\tilde{p}; \bar{\omega}(s))$ denote the fraction of informed agents who buy at p and $\tilde{p}$, respectively. In period 1, $d_1 = D^I(p; \bar{\omega}(s)) = d^I$. The aggregate biased belief in period 2 is $\bar{\omega}_2 = (1-\alpha)\bar{\omega}(s) + \alpha p$, and Equation (11) then implies that the fraction of uninformed agents who buy in period 2 is $D^U(p; \bar{\omega}_2) = d_I$. Thus, the overall fraction of agents who buy in period 2 is $d_2 = d^I$. Equation (C.2) then implies that $\bar{\omega}_3 = \bar{\omega}_2$. Hence, if $n^* \geq 3$, then $d_3 = d_2 = d^I$. It is straightforward to see that this logic giving rise to a constant aggregate biased belief and quantity demanded will continue until the first period with the new price, $\tilde{p}$.

Step 2: Quantity demanded increases beyond the rational benchmark when the price drops. Since the quantity demanded is constant prior to the price change, we can (without loss of generality) assume from now on that $n^* = 1$. That is, $p_1 = p$ and $p_n = \tilde{p}$ for all $n \geq 2$. In all periods $n \geq 2$, the fraction of informed agents who buy is $\tilde{d}^I$. By contrast, in period 2, the fraction of uninformed agents who buy is $\tilde{d}_2^U \equiv D^U(\tilde{p}; \bar{\omega}_2) = 1 - F\left(\frac{\tilde{p}-\bar{\omega}_2}{1-\alpha}\right)$. Importantly, $\tilde{d}_2^U > \tilde{d}^I$. To see this, note that $\bar{\omega}_2 = (1-\alpha)\bar{\omega}(s) + \alpha p$ and hence

$$\begin{aligned} \tilde{d}_2^U &= 1 - F\left(\frac{\tilde{p} - (1-\alpha)\bar{\omega}(s) - \alpha p}{1-\alpha}\right) \\ &= 1 - F\left(\tilde{p} - \bar{\omega}(s) - \frac{\alpha}{1-\alpha}(p - \tilde{p})\right) > 1 - F(\tilde{p} - \bar{\omega}(s)) = \tilde{d}^I, \end{aligned} \tag{C.8}$$

where the inequality follows from $p - \tilde{p} > 0$. Thus, the total quantity demanded in period 2 is $d_2 = \lambda\tilde{d}^I + (1-\lambda)\tilde{d}_2^U$, which exceeds the rational benchmark of $\tilde{d}^I$.

Step 3: Quantity demanded remains above the rational benchmark in all subsequent periods. We now consider the path of $\tilde{d}_n^U \equiv D^U(\tilde{p}; \bar{\omega}_n) = 1 - F\left(\frac{\tilde{p}-\bar{\omega}_n}{1-\alpha}\right)$ for $n > 2$ starting from the initial condition of $\tilde{d}_2^U = 1 - F\left(\frac{\tilde{p}-\bar{\omega}_2}{1-\alpha}\right)$. From the law of motion in Equation (C.2), we must have that for all $n \geq 2$,

$$\tilde{d}_{n+1}^U = D^U(\tilde{p}; \bar{\omega}_{n+1}) = \lambda\tilde{d}_I + (1-\lambda)\tilde{d}_n^U. \tag{C.9}$$

Thus, if $\tilde{d}_n^U > \tilde{d}^I$, then $\tilde{d}_{n+1}^U > \tilde{d}^I$. Since we start from the base case of $\tilde{d}_2^U > \tilde{d}^I$, induction on n implies that $\tilde{d}_n^U > \tilde{d}^I$ for all $n \geq 2$. Thus, the aggregate quantity demanded in any period $n \geq 2$ is $d_n = \lambda\tilde{d}^I + (1-\lambda)\tilde{d}_n^U > \tilde{d}^I$, and d_n therefore exceeds the rational benchmark.

Part 2. We now show that d_n converges to the rational benchmark level of $\tilde{d}^I$ as $n \to \infty$. Toward this end, we first show that for all $k \geq 1$,

$$\tilde{d}_{k+2}^U = \left[1 - (1-\lambda)^k\right]\tilde{d}^I + (1-\lambda)^k\tilde{d}_2^U. \tag{C.10}$$

We will show by induction that in each period $k+2$, we have $\tilde{d}^U_{k+2} = a_{k+2}\tilde{d}^I + b_{k+2}\tilde{d}^U_2$, and that the coefficients a_{k+2} and b_{k+2} satisfy $a_{k+2} + b_{k+2} = 1$ and $b_{k+2} = (1-\lambda)^k$. The base case ($k = 1$) is immediate from (C.9), since $\tilde{d}^U_3 = \lambda\tilde{d}_I + (1-\lambda)\tilde{d}^U_2$ For the induction step, suppose the claim is true for $k > 1$. Thus, $\tilde{d}^U_{k+2} = a_{k+2}\tilde{d}^I + b_{k+2}\tilde{d}^U_2$. From (C.9), this implies that

$$\tilde{d}^U_{k+3} = \lambda\tilde{d}^I + (1-\lambda)[a_{k+2}D^I + b_{k+2}\tilde{d}^U_2] = \underbrace{[\lambda + (1-\lambda)a_{k+2}]}_{\equiv a_{k+3}}\tilde{d}^I + \underbrace{(1-\lambda)b_{k+2}}_{\equiv b_{k+3}}\tilde{d}^U_2. \tag{C.11}$$

It is then immediate that $b_{k+3} = (1-\lambda)^{k+1}$ as required given the induction assumption of $b_{k+2} = (1-\lambda)^k$. To show that $a_{k+3} + b_{k+3} = 1$, note that $a_{k+2} + b_{k+2} = 1$ implies

$$a_{k+3} + b_{k+3} = \lambda + (1-\lambda)a_{k+2} + (1-\lambda)b_{k+2} = \lambda + (1-\lambda)[a_{k+2} + b_{k+2}] = 1. \tag{C.12}$$

The deviation between the quantity demanded in period n under projection and the rational benchmark quantity is $|d_n - \tilde{d}^I| = |\lambda\tilde{d}^I + (1-\lambda)\tilde{d}^U_n - \tilde{d}^I| = (1-\lambda)|\tilde{d}^U_n - \tilde{d}^I|$, and (C.10) implies that for $n \geq 2$, $|\tilde{d}^U_n - \tilde{d}^I| = (1-\lambda)^{n-2}|\tilde{d}^U_2 - \tilde{d}^I|$. Thus,

$$|d_n - \tilde{d}^I| = (1-\lambda)^{n-1}|\tilde{d}^U_2 - \tilde{d}^I|. \tag{C.13}$$

This value is clearly decreasing in n and converges to 0 as $n \to \infty$. Thus, d_n converges to the rational benchmark, $\tilde{d}^I$, as $n \to \infty$. ■

Proof of Lemma C.1. As noted in the text, we restrict attention to the case in which it is never optimal to serve the lowest type. In this case, Equation (C.1) implies that the true demand function in period $n \geq 2$ is $D(p_n; \bar{\omega}_n, \bar{\omega}(s)) = \lambda D^I(p_n; \bar{\omega}(s)) + (1-\lambda)D^U(p_n; \bar{\omega}_n)$, where D^I and D^U are specified in Equation (14). Hence,

$$D(p_n; \bar{\omega}_n, \bar{\omega}(s)) = \frac{(1-\alpha)\bar{t} + \lambda(1-\alpha)\bar{\omega}(s) + (1-\lambda)\bar{\omega}_n - (1-\lambda\alpha)p_n}{(1-\alpha)(\bar{t} - \underline{t})}. \tag{C.14}$$

In period $n+1$, an uninformed observer with taste t thinks that when the preceding generation holds a common expectation of ω equal to $\hat{\omega}$, then their demand is given by

$$\widehat{D}(p_n; \hat{\omega}|t) = \frac{(1-\alpha)\bar{t} + \hat{\omega} - p_n + \alpha t}{(1-\alpha)(\bar{t} - \underline{t})}. \tag{C.15}$$

The inferred value of this observer, denoted $\hat{\omega}_{n+1}(t)$, is the value of $\hat{\omega}$ that solves $\widehat{D}(p_n; \hat{\omega}|t) = D(p_n; \bar{\omega}_n, \bar{\omega}(s))$. By Proposition 6, Part 1, $\hat{\omega}_{n+1}(t) = \bar{\omega}_{n+1} - \alpha t$. Substituting this into the previous equality and solving for $\bar{\omega}_{n+1}$ in terms of $\bar{\omega}_n$ yields the following law of motion:

$$\bar{\omega}_{n+1} = \lambda\big[(1-\alpha)\bar{\omega}(s) + \alpha p_n\big] + (1-\lambda)\bar{\omega}_n, \tag{C.16}$$

starting from $\bar{\omega}_2 = (1-\alpha)\bar{\omega}(s) + \alpha p_1$. We complete the proof using induction on $n \geq 2$. Define

$$\tilde{p}^{n-1} \equiv (1-\lambda)^{n-2} p_1 + \sum_{k=2}^{n-1} \lambda(1-\lambda)^{n-1-k} p_k. \tag{C.17}$$

For the base case, note that (C.16) implies that $\bar{\omega}_3 = \lambda\big[(1-\alpha)\bar{\omega}(s)+\alpha p_2\big] + (1-\lambda)[(1-\alpha)\bar{\omega}(s) + \alpha p_1] = (1-\alpha)\bar{\omega}(s) + \alpha[(1-\lambda)p_1 + \lambda p_2] = (1-\alpha)\bar{\omega}(s) + \alpha\tilde{p}^2$. Now suppose that for any $n > 2$, we have $\bar{\omega}_n = (1-\alpha)\bar{\omega}(s) + \alpha\tilde{p}^{n-1}$. Again, (C.16) implies that $\bar{\omega}_{n+1} = \lambda\big[(1-\alpha)\bar{\omega}(s) + \alpha p_n\big] + (1-\lambda)[(1-\alpha)\bar{\omega}(s) + \alpha\tilde{p}^{n-1}] = (1-\alpha)\bar{\omega}(s) + \alpha[(1-\lambda)\tilde{p}^{n-1} + \lambda p_n] = (1-\alpha)\bar{\omega}(s) + \alpha\tilde{p}^n$. ■

Proof of Proposition C.2. As noted in the text, we restrict attention to the case in which it is never optimal to serve the lowest type. Thus, the optimal price path is characterized by the first-order conditions, aside from the possibility of pricing at the ceiling. We discuss the price-ceiling case at the end of the proof and focus on the interior case first. In the interior case, profit in period $n \geq 2$ is

$$\Pi(p_n;\bar{\omega}_n,\bar{\omega}(s)) = p_n D(p_n;\bar{\omega}_n,\bar{\omega}(s)) = p_n\left(\frac{(1-\alpha)\bar{t} + \lambda(1-\alpha)\bar{\omega}(s) + (1-\lambda)\bar{\omega}_n - (1-\lambda\alpha)p_n}{(1-\alpha)(\bar{t}-\underline{t})}\right);$$

in period $n=1$, profit is $\widetilde{\Pi}(p_1;\bar{\omega}(s)) = p_1\left(\frac{\bar{t}+\bar{\omega}_1-p_1}{\bar{t}-\underline{t}}\right)$. The seller's maximization problem is thus

$$\max_{\{p_n\}_{n=1}^N}\left(\widetilde{\Pi}(p_1;\bar{\omega}(s)) + \sum_{n=2}^N \Pi(p_n;\bar{\omega}_n)\right) \quad \text{s.t.} \quad \bar{\omega}_{n+1} = \varphi(\bar{\omega}_n,p_n)\ \forall n = 2,\ldots,N, \tag{C.18}$$

where $\varphi(\bar{\omega}_n;p_n) \equiv \lambda\big[(1-\alpha)\bar{\omega}(s) + \alpha p_n\big] + (1-\lambda)\bar{\omega}_n$ is the transition function derived in Lemma C.1. The Lagrangian is then

$$\mathcal{L} = \widetilde{\Pi}(p_1;\bar{\omega}_1) + \sum_{n=2}^N \Pi(p_n;\bar{\omega}_n) + \sum_{n=1}^N \gamma_n(\bar{\omega}_{n+1} - \varphi(\bar{\omega}_n,p_n)), \tag{C.19}$$

where $\{\gamma_n\}_{n=1}^N$ are Lagrange multipliers.

The plan for the proof is to first develop a set of equations (first-order conditions and Euler equations) that characterize the optimal price path. We will then argue that the price in the final period, p_N, must be lower than p_{N-1} by the same logic underlying the two-period case (Proposition 4). We then argue by induction that if for any n we have $p_n > p_{n+1} > \cdots > p_N$, then $p_{n-1} > p_n$, which establishes the declining price path (i.e. Part 2 of the proposition). Finally, we will note that

$p_1 > p_M$ (i.e,. Part 1).

We begin by deriving a set of first-order conditions that characterize the system of prices. Given the functional forms of Π, $\widetilde{\Pi}$, and φ, we have the following collection of first-order conditions: (i) the FOC w.r.t. p_1 is

$$\frac{\bar{t} + \bar{\omega}(s) - 2p_1}{\bar{t} - \underline{t}} = \gamma_1 \alpha; \tag{C.20}$$

(ii) the FOC w.r.t. p_n for $n = 2, \ldots, N-1$ is

$$\left(\frac{(1-\alpha)\bar{t} + \lambda(1-\alpha)\bar{\omega}(s) + (1-\lambda)\bar{\omega}_n - 2(1-\lambda\alpha)p_n}{(1-\alpha)(\bar{t} - \underline{t})} \right) = \gamma_n \lambda \alpha; \tag{C.21}$$

(iii) the FOC w.r.t. p_N is

$$\left(\frac{(1-\alpha)\bar{t} + \lambda(1-\alpha)\bar{\omega}(s) + (1-\lambda)\bar{\omega}_N - 2(1-\lambda\alpha)p_N}{(1-\alpha)(\bar{t} - \underline{t})} \right) = 0, \tag{C.22}$$

which follows from the fact that $\gamma_N = 0$ given the FOC w.r.t. to $\bar{\omega}_{N+1}$; and (iv) the FOC w.r.t. $\bar{\omega}_n$ for $n = 2, \ldots, N$ is

$$p_n \left(\frac{1-\lambda}{(1-\alpha)(\bar{t} - \underline{t})} \right) + \gamma_{n-1} = \gamma_n (1-\lambda). \tag{C.23}$$

From these FOCs, we can derive an "Euler equation" by using the FOC for p_{n-1} in (C.21) to solve for γ_{n-1} and then substituting this value into (C.23). The result provides a link between p_{n-1} and p_n in terms of the current beliefs. Equations (C.20) and (C.23)—along with the fact that $p^M = (\bar{t} + \bar{\omega}(s))/2$—imply that the Euler equation linking periods 1 and 2 is

$$p_2 = \left(\frac{2\lambda(1-\alpha) + \alpha(1-\lambda)^2}{(1-\lambda)(2-\lambda\alpha)} \right) p_1 - \frac{2(2\lambda-1)(1-\alpha)}{(1-\lambda)(2-\lambda\alpha)} p^M. \tag{C.24}$$

For $n > 2$, equations (C.21) and (C.23) along with the expression for $\bar{\omega}_n$ in terms of past prices (from Lemma C.1) imply that the Euler equation linking periods $n-1$ and n is:

$$p_n = \phi_{-1} p_{n-1} - \phi_M p^M - \tilde{\phi} \tilde{p}^{n-2} \tag{C.25}$$

where we've introduced the following positive constants:

$$\begin{aligned}
\phi_{-1} &= \frac{(2-\alpha\lambda) - \alpha\lambda^2(2-\lambda)}{(1-\lambda)(2-\alpha\lambda)}, && \text{(C.26)}\\
\phi_M &= \frac{2\lambda(1-\alpha)}{(1-\lambda)(2-\lambda\alpha)}, && \text{(C.27)}\\
\tilde{\phi} &= \alpha \frac{\lambda(2-\lambda)}{(2-\lambda\alpha)}. && \text{(C.28)}
\end{aligned}$$

To characterize the solution, we will combine these Euler equations with the FOCs for each p_n. Using our expression for $\bar{\omega}_n$ in terms of past prices (from Lemma C.1), the FOCs w.r.t. p_n for $n \geq 2$ from above can be equivalently written as

$$\begin{aligned} 0 &= (1-\alpha)(\bar{t}+\bar{\omega}(s)) + \alpha(1-\lambda)\tilde{p}^{n-1} - 2(1-\lambda\alpha)p_n + \alpha(1-\lambda)\sum_{k=n+1}^{N} p_k \frac{\partial \tilde{p}^{k-1}}{\partial p_n} \\ &= 2(1-\alpha)p^M + \alpha(1-\lambda)\tilde{p}^{n-1} - 2(1-\lambda\alpha)p_n + \alpha\lambda \sum_{k=n+1}^{N}(1-\lambda)^{k-n}p_k, \end{aligned} \tag{C.29}$$

where we've used the fact that $\frac{\partial \tilde{p}^{k-1}}{\partial p_n} = \lambda(1-\lambda)^{k-n-1}$ and $p^M = (\bar{t}+\bar{\omega}(s))/2$ in the uniform case. Given that the demand function in period 1 is different from the one in $n \geq 2$, the FOC w.r.t. p_1 is

$$0 = (1-\alpha)p^M - 2(1-\alpha)p_1 + \alpha\sum_{k=2}^{N}(1-\lambda)^{k-1}p_k \tag{C.30}$$

since $\frac{\partial \tilde{p}^{k-1}}{\partial p_1} = (1-\lambda)^{k-2}$. To summarize, the N prices must solve the following system of N equations:

$$\begin{aligned} p_1 &= p^M + \frac{\alpha}{2(1-\alpha)}\left(\sum_{k=2}^{N}(1-\lambda)^{k-1}p_k\right) \\ &\vdots \\ p_n &= \left(\frac{1-\alpha}{1-\lambda\alpha}\right)p^M + \left(\frac{\alpha}{2(1-\lambda\alpha)}\right)\left((1-\lambda)\tilde{p}^{n-1} + \lambda\sum_{k=n+1}^{N}(1-\lambda)^{k-n}p_k\right) \\ &\vdots \\ p_N &= \left(\frac{1-\alpha}{1-\lambda\alpha}\right)p^M + \left(\frac{\alpha}{2(1-\lambda\alpha)}\right)\left((1-\lambda)\tilde{p}^{N-1}\right). \end{aligned} \tag{C.31}$$

Going forward, we will streamline notation by letting $c_n \equiv p_n/p^M$ denote the "normalized" price in each period n. This allows us to characterize the system for $(c_1, \ldots, c_N)$ without any explicit dependence on the value of p^M. Similarly, for all n, let $\tilde{c}^{n-1} = \tilde{p}^{n-1}/p^M = (1-\lambda)^{n-2}c_1 + \sum_{k=2}^{n-1}\lambda(1-\lambda)^{n-1-k}c_k$. Additionally, let $\hat{c}^{n+1} \equiv \sum_{k=n+1}^{N}(1-\lambda)^{k-n}p_k/p^M = \sum_{k=n+1}^{N}(1-\lambda)^{k-n}c_k$.

We now prove the following via induction: for $n > 2$, if $c_n > c_{n+1} > \cdots > c_N$, then $c_{n-1} > c_n$.

Base Case: $c_{N-1} > c_N$. We prove the base case by showing $c_{N-1} > c_N$. From (C.29), the FOC w.r.t. c_{N-1} is $2(1-\alpha) + \alpha(1-\lambda)\tilde{c}^{N-2} - 2(1-\lambda\alpha)c_{N-1} + \alpha\lambda(1-\lambda)c_N = 0$, and the FOC w.r.t. c_N is $2(1-\alpha) + \alpha(1-\lambda)\tilde{c}^{N-1} - 2(1-\lambda\alpha)c_N = 0$. The definition of $\tilde{c}^{N-1}$ implies that that $\tilde{c}^{N-1} = (1-\lambda)\tilde{c}^{N-2} + \lambda c_{N-1}$. Substituting this value into the latter FOC and equating the two

FOCs yields the following necessary condition:

$$\alpha\lambda(1-\lambda)\tilde{c}^{N-2} = \Big(2(1-\lambda\alpha) + \alpha\lambda(1-\lambda)\Big)[c_{N-1} - c_N]. \tag{C.32}$$

It is straightforward to verify that $2(1-\lambda\alpha) + \alpha\lambda(1-\lambda) = 2 - \alpha\lambda[1+\lambda] > 0$ for any $\alpha \in (0,1)$ and any $\lambda \in (0,1)$. Thus, since the left-hand side of (C.32) is strictly positive (it is a weighted sum of normalized prices), we have $c_{N-1} > c_N$.

Induction step: $c_n > c_{n+1}$ for $n \geq 2$. Consider $n \in \{3, \ldots, N-1\}$ and suppose that $c_n > c_{n+1} > \cdots > c_N$. We will show that $c_{n-1} > c_n$. To do so, we first derive an expression for c_{n-1} purely in terms of $(c_n, \ldots, c_N)$. Note that neither the Euler equation for c_{n-1} nor the FOC w.r.t. c_{n-1} provides this: the former characterizes c_{n-1} as a function of previous prices, $(c_1, \ldots, c_{n-1})$ and the latter characterizes c_{n-1} as a function of previous *and* future prices. To obtain this expression, note that (C.25) implies $\tilde{c}^{n-2} = (\phi_{-1}c_{n-1} - c_n - \phi_M)/\tilde{\phi}$. Substituting this value into the FOC w.r.t. c_{n-1} (Equation C.29) yields

$$2(1-\lambda\alpha)c_{n-1} = 2(1-\alpha) + \alpha(1-\lambda)\frac{1}{\tilde{\phi}}\Big(\phi_{-1}c_{n-1} - c_n - \phi_M\Big) + \alpha\lambda\hat{c}^n. \tag{C.33}$$

From the definition of $\hat{c}^n$, note that $\hat{c}^n = (1-\lambda)c_n + (1-\lambda)\hat{c}^{n+1}$. Substituting this expression into (C.33) and substituting the values of the constants ϕ_{-1}, ϕ_M, and $\tilde{\phi}$ from above (Equations C.26 to C.28), and simplifying, reveals that

$$c_{n-1} = \phi_{-1}c_n + \phi_M - \left(\frac{\lambda}{1-\lambda}\right)\tilde{\phi}\hat{c}^{n+1}. \tag{C.34}$$

Recall that, by assumption, $c_n > c_{n+1} > \cdots > c_N$, and we want to show $c_{n-1} > c_n$. From (C.34), this condition is equivalent to $\phi_{-1}c_n + \phi_M - \left(\frac{\lambda}{1-\lambda}\right)\tilde{\phi}\hat{c}^{n+1} > c_n$, and hence equivalent to

$$[\phi_{-1} - 1]c_n > \left(\frac{\lambda}{1-\lambda}\right)\tilde{\phi}\hat{c}^{n+1} - \phi_M. \tag{C.35}$$

From the definition of ϕ_{-1}, we have $\phi_{-1} - 1 > 0$. Notice that (C.34) must hold for all $n \in \{3, \ldots, N-1\}$, and hence $c_n = \phi_{-1}c_{n+1} + \phi_M - \left(\frac{\lambda}{1-\lambda}\right)\tilde{\phi}\hat{c}^{n+2}$. Moreover, note that the definitions of ϕ_{-1} and $\tilde{\phi}$ are such that $\phi_{-1} = (1-\lambda\tilde{\phi})/(1-\lambda)$; substituting this into the previous equality along with the fact that $\hat{c}^{n+1} = (1-\lambda)c_{n+1} + (1-\lambda)\hat{c}^{n+1}$ implies that $\left(\frac{\lambda}{1-\lambda}\right)\tilde{\phi}\hat{c}^{n+1} = -(1-\lambda)c_n + (1-\lambda)\phi_M + c_{n+1}$. Substituting this into the inequality of interest (Condition C.35) yields the equivalent condition of $[\phi_{-1} - \lambda]c_n > c_{n+1} - \lambda\phi_M$. Since we know $c_n > c_{n+1}$ and since $\phi_{-1} - \lambda > 0$ (because $\phi_{-1} > 1$, as noted above), the previous condition will hold at $c_n > c_{n+1}$ if it holds at $c_n = c_{n+1}$. Thus, it suffices to show that $[\phi_{-1} - \lambda]c_{n+1} > c_{n+1} - \lambda\phi_M \Leftrightarrow [\phi_{-1} - \lambda - 1]c_{n+1} > -\lambda\phi_M$. The previous condition holds so long as $\phi_{-1} - \lambda - 1 > 0$, which can be directly confirmed from the definition of ϕ_{-1} in (C.26). This completes the induction step.

So far, we have verified that $c_{N-1} > c_N$ implies that $c_n > c_{n+1}$ for all $n \geq 2$. To complete the proof, we must show that $c_2 > c_3 > \cdots > c_N$ implies that $c_1 > c_2$. Since the Euler equation linking periods 1 and 2 is different from one in all other periods, we cannot rely on (C.34) as above. Instead, consider the FOCs in periods 1 and 2 (Equations C.30 and C.29), which are $2(1-\alpha) - 2(1-\alpha)c_1 + \alpha\hat{c}^2 = 0$ and $2(1-\alpha) + \alpha(1-\lambda)\tilde{c}^1 - 2(1-\lambda\alpha)c_2 + \alpha\lambda\hat{c}^3 = 0$, respectively. Using the fact that $\hat{c}^2 = (1-\lambda)c_2 + (1-\lambda)\hat{c}^3$, equating the two FOCs and simplifying yields the condition

$$\alpha[(1-2\lambda)\hat{c}^2 + 2(1-\lambda)c_2] = \zeta[c_1 - c_2], \tag{C.36}$$

where $\zeta = [2(1-\alpha) + \alpha(1-\lambda)] = 2 - \alpha(1+\lambda)$; note that $\zeta \in (0, 2)$ for all $\alpha \in (0, 1)$. Thus, we have $c_1 > c_2$ so long as $(1-2\lambda)\hat{c}^2 + 2(1-\lambda)c_2 > 0 \Leftrightarrow 2(1-\lambda)c_2 > (2\lambda - 1)\hat{c}^3$. While this holds immediately whenever $\lambda < 1/2$, we must show it holds more generally. Recall that $\hat{c}^3 = \sum_{k=3}^{N}(1-\lambda)^{k-2}c_k$. Substituting this into the previous inequality yields the equivalent condition of $2(1-\lambda)c_2 > (2\lambda-1)\sum_{k=3}^{N}(1-\lambda)^{k-2}c_k \Leftrightarrow 2c_2 > (2\lambda-1)\sum_{k=3}^{N}(1-\lambda)^{k-3}c_k$. Since we've assumed $c_2 > c_3 > \cdots > c_N$, a sufficient condition for the previous inequality is

$$2c_2 > (2\lambda-1)c_2\sum_{k=3}^{N}(1-\lambda)^{k-3} \Leftrightarrow 2 > (2\lambda-1)\sum_{k=0}^{N-3}(1-\lambda)^k. \tag{C.37}$$

Recall that the partial sum of the geometric series is $\sum_{k=0}^{N-3}(1-\lambda)^k$ is strictly less than $\frac{1}{1-(1-\lambda)} = \frac{1}{\lambda}$. Thus, a sufficient condition for Condition (C.37) is $2 > (2\lambda-1)\frac{1}{\lambda}$, which necessarily holds.

Finally, it is immediate from the FOC for p_1 in (C.29) that $p_1 > p^M$. Similarly, if the FOC in period 1 does not hold because the seller prefers setting p_1 equal to the price ceiling, $\bar{p}$, then the logic of this proof remains unchanged. If $p_1 = \bar{p}$, then clearly we have $p_1 > p^M$; moreover, the seller would never charge $p_2 = \bar{p}$ if $p_1 = \bar{p}$ since she strictly profits from a price decrease in period 2. Thus, it is immediate that we still have $p_2 < p_1 = \bar{p}$ in this case, and hence prices will follow the interior path described above from period 2 onward. ■

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